Statistics for SPDEs

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How to get into statistics for SPDEs?

- S. V. Lototsky and B. L. Rozovsky (2017) Stochastic partial differential equations, *Springer*.
- **I. Cialenco (2018) Statistical inference for SPDEs: an overview, *SISP*.**
- **stats4SPDEs:** `sites.google.com/view/stats4spdes`

Cell motility: repolarisation due to signal gradient

Meinhardt Model

$$\frac{\partial A}{\partial t} = \frac{s r_a (\frac{A^2}{B} + b_a)}{(s_c + C)(1 + s_a A^2)} - r_a A + D_A \frac{\partial^2 A}{\partial x^2}$$

$$\frac{dB}{dt} = r_b \sum_n \frac{A}{n} - r_b B$$

$$\frac{\partial C}{\partial t} = b_c A - r_c C + D_C \frac{\partial^2 C}{\partial x^2}$$

Levchenko Model

$$\frac{\partial A}{\partial t} = k_A s - k_{-A} A$$

$$\frac{\partial I}{\partial t} = k_I s - k_{-I} I + D_I \frac{\partial^2 I}{\partial x^2}$$

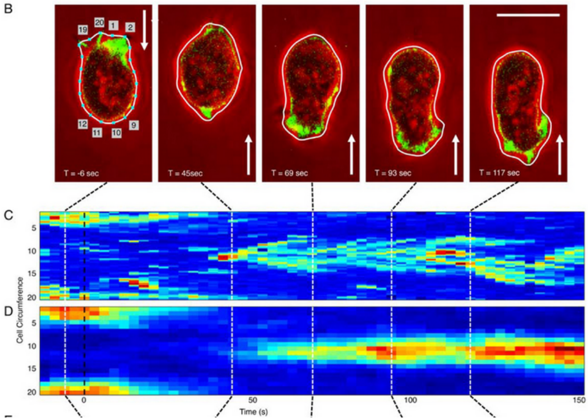
$$\frac{\partial R}{\partial t} = k_R A (R_T - R) - k_{-R} I R$$

Otsuji Model

$$\frac{\partial U}{\partial t} = a_1 \left[V - \frac{U + V}{(a_2 s (U + V) + 1)^2} \right] + D_U \frac{\partial^2 U}{\partial x^2}$$

$$\frac{\partial V}{\partial t} = a_1 \left[\frac{U + V}{(a_2 s (U + V) + 1)^2} - V \right] + D_V \frac{\partial^2 V}{\partial x^2}$$

$$s = (1 + dy\cos(2\pi x))$$



Lockley, Ladds, Bretschneider (2015) Image based validation of dynamical models for cell reorientation, *Cytometry A*

A stochastic Meinhardt model

Activator-inhibitor model $X = (A, I)$:

$$\begin{cases} \partial_t A(t,x) = D_A \partial_x^2 A(t,x) + f_A(X(t,x),x) + \sigma_A \dot{W}_A(t,x), \\ \partial_t I(t,x) = D_I \partial_x^2 I(t,x) + f_I(X(t,x),x) + \sigma_I \dot{W}_I(t,x), \end{cases}$$

on the torus $\mathbb{R}/L\mathbb{Z}$ with *space-time white noises* $\dot{W}_A$, $\dot{W}_I$,

$$f_A(u,x) = r_A \frac{s(x)\left(b_A + u_1^2\right)}{(s_I + |u_2|)\left(1 + s_A u_1^2\right)} - r_A u_1, \quad f_I(u,x) = b_I u_1 - r_I u_2,$$

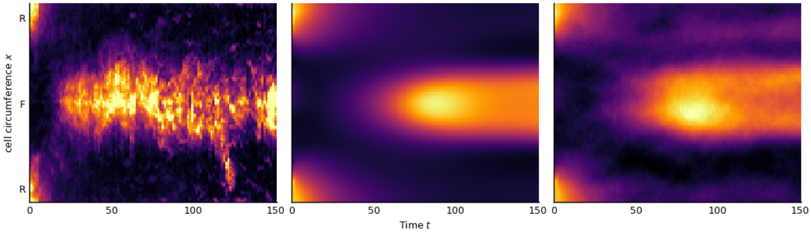
and extracellular signal $s(x)$.

Observation: $A(t, x_i)$ for $t \in [0, T]$, locations x_i

Goal: estimate unknown parameters like diffusivity D_A

Altmeyer, Bretschneider, Janak, Reiß (2022): Parameter estimation in an SPDE-model for cell repolarisation

Cell repolarisation: data and models



Heat maps for the space-time evolution of the activator A

Left: experimental data averaged over 18 cells

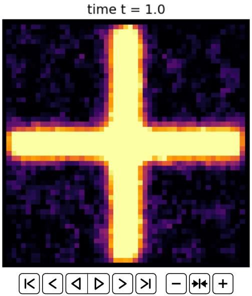
Center: solution to the deterministic Meinhardt model

Right: stochastic Meinhardt model with noise level 0.02

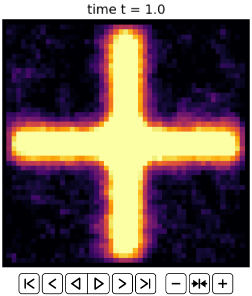
See
https://www.mathematik.hu-berlin.de/de/forschung/forschungsgebiete/stochastik/stoch-employees/mreiss/publications

Stochastic heat equation: simulations

$$\dot{X}(t,x) = \vartheta \Delta X(t,x) + \dot{W}(t,x)$$



Left: $\vartheta = 4$



Right: $\vartheta = 8$

Outline

Drift estimation: Girsanov Theorem

Observe $(X_t,\ t \in [0,T])$ continuously in time, where

$$dX(t) = b_\vartheta(X(t))\,dt + \sigma(X(t))\,dW(t), \quad t \in [0,T]$$

with $X(0) = x_0$, $b_\vartheta, \sigma : \mathbb{R} \to \mathbb{R}$ regular, parameter $\vartheta \in \Theta$ and 1D-Brownian motion W.

Likelihood via Girsanov:(Liptser, Shiryaev 2001)

$$L(\vartheta) = \frac{d\,\mathbb{P}^X_\vartheta}{d\,\mathbb{P}^{\sigma W}} = \exp\Big(\int_0^T \frac{b_\vartheta(X(t))}{\sigma(X(t))^2}\,dX(t) - \frac{1}{2}\int_0^T \frac{b_\vartheta(X(t))^2}{\sigma(X(t))^2}\,dt\Big)$$

Maximum-likelihood estimator (MLE):

$$\hat{\vartheta} \in \operatorname{argmax}_{\vartheta\in\Theta}\Big(\int_0^T \frac{b_\vartheta(X(t))}{\sigma(X(t))^2}\,dX(t) - \frac{1}{2}\int_0^T \frac{b_\vartheta(X(t))^2}{\sigma(X(t))^2}\,dt\Big)$$

Volatility estimation: identifiability

Observe $(X_t,\ t \in [0,T])$ continuously in time, where

$$dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t), \quad t \in [0,T]$$

Quadratic variation for $t \in (0,T]$:

$$\langle X \rangle_t = \lim_{n\to\infty} \sum_{i=1}^{n} (X(ti/n) - X(t(i-1)/n))^2 = \int_0^t \sigma(X(t))^2 dt$$

Consequence: $\frac{d}{dt}\langle X \rangle_t = \sigma^2(X(t))$ implies $\sigma^2(\bullet)$ is *identifiable* on $\{X(t) \mid t \in [0,T]\} \subseteq \mathbb{R}$.

Discrete observations: Let $0 = t_0 < t_1 < \cdots < t_n = T$. Observing $X(t_i)$, use for $h > 0$ the estimator

$$\hat\sigma^2_{n,h}(x) = \frac{\sum_{i:|X(t_{i-1})-x|\leqslant h} (X(t_i) - X(t_{i-1}))^2}{\#\{i : |X(t_{i-1}) - x| \leqslant h\}}$$

MLE for Ornstein-Uhlenbeck process

Observe $(X_t,\ t \in [0, T])$ continuously in time, where

$$dX(t) = \vartheta X(t)dt + \sigma dW(t)$$

Maximum-likelihood estimator (MLE):

$$\hat{\vartheta} := \frac{\int_0^T X(t)dX(t)}{\int_0^T X_t^2 dt}$$

$$= \frac{\int_0^T X(t)\left(\vartheta X(t)\,dt + \sigma\,dW(t)\right)}{\int_0^T X(t)^2 dt} = \vartheta + \frac{\int_0^T X(t)\sigma\,dW(t)}{\int_0^T X(t)^2 dt}$$

Asymptotic theory for $\vartheta < 0$:(Kutoyants 2004)

$$\frac{\sqrt{T}}{\sqrt{2|\vartheta|}}\left(\hat{\vartheta} - \vartheta\right) \xrightarrow{T|\vartheta|\to\infty} \mathcal{N}(0,1)$$

Asymptotics of MLE: proof

$$\hat{\vartheta} - \vartheta = \frac{\int_0^T X(t)\sigma\, dW(t)}{\int_0^T X(t)^2 dt} = \frac{\sigma M(T)}{\langle M\rangle_T}$$

The *observed Fisher information* $\langle M\rangle_T$ satisfies (for $X(0) = 0$)

$$\mathbb{E}[\langle M\rangle_T] = \int_0^T \mathbb{E}[X(t)^2]\, dt = \int_0^T \frac{(1-e^{2\vartheta t})\sigma^2}{2|\vartheta|}\, dt = \frac{T\sigma^2}{2|\vartheta|}(1 + o(1))$$

and $\mathrm{Var}(\langle M\rangle_T) = o(\mathbb{E}[\langle M\rangle_T]^2)$. Hence, $\frac{2|\vartheta|}{T\sigma^2}\langle M\rangle_T \xrightarrow{\mathbb{P}} 1$ and a martingale CLT yields $\frac{\sqrt{T}\sigma}{\sqrt{2|\vartheta|}}\frac{M(T)}{\langle M\rangle_T} \xrightarrow{d} \mathcal{N}(0, 1)$. □

Lower bound: Fisher information $I(\vartheta) = \mathbb{E}[\langle M\rangle_T]$ and LAN
$\Rightarrow$ no estimator can have smaller asymptotic variance.

Estimation of reaction term in SPDE

Likelihood theory for SPDE drift estimation?

Likelihood for SPDEs

Stochastic reaction-diffusion equation:

$$dX(t) = (\nu\Delta X(t) + F_\vartheta(X(t)))\,dt + B\,dW(t), \quad t \in [0, T]$$

with $B : L^2(\Lambda) \to L^2(\Lambda)$ linear, invertible and $X(0) = x_0$.

Girsanov Theorem: with $F_0 = 0$ (Da Prato, Zabczyk 2014)

$$L(\vartheta) = \frac{d\,\mathbb{P}^X_\vartheta}{d\,\mathbb{P}^X_0} = \exp\Big(\int_0^T \langle B^{-1}F_\vartheta(X(t)), B^{-1}(dX(t) - \nu\Delta X(t)dt)\rangle_{L^2(\Lambda)} - \frac{1}{2}\int_0^T \|B^{-1}F_\vartheta(X(t))\|^2_{L^2(\Lambda)}dt\Big)$$

MLE: ($F_\vartheta = \vartheta F$, $\vartheta \in \mathbb{R}$)

$$\hat{\vartheta} = \frac{\int_0^T \langle B^{-1}F(X(t)), B^{-1}(dX(t) - \nu\Delta X(t)dt)\rangle_{L^2(\Lambda)}}{\int_0^T \|B^{-1}F(X(t))\|^2_{L^2(\Lambda)}dt}$$

MLE for reaction parameter

1D-Stochastic reaction-diffusion equation:

$$dX(t,x) = \left(\nu\Delta X(t,x) + \vartheta f(X(t,x))\right) dt + \sigma\, dW(t), \quad t \in [0,T]$$

MLE (under $\mathbb{P}^X_\vartheta$): $\hat{\vartheta}_\nu = \vartheta + \frac{\int_0^T \int_\Lambda f(X(t,x))\sigma\, dW(t,x)}{\int_0^T \int_\Lambda f(X(t,x))^2 dxdt}$.

Consistency if $\int_0^T \int_\Lambda \frac{f(X(t,x))^2}{\sigma^2} dxdt \to \infty$.

Small diffusivity asymptotics: $\nu \to 0$, σ, T fixed (Gaudlitz, MR 2023)

$$\hat{\vartheta}_\nu - \vartheta = \mathcal{O}_{\mathbb{P}}\left(\left(\int_0^T \|e^{t\nu\Delta}\|^2_{HS} dt\right)^{-1/2}\right) = \mathcal{O}_{\mathbb{P}}(\nu^{1/4})$$

if f has linear growth. This is optimal (LAN).

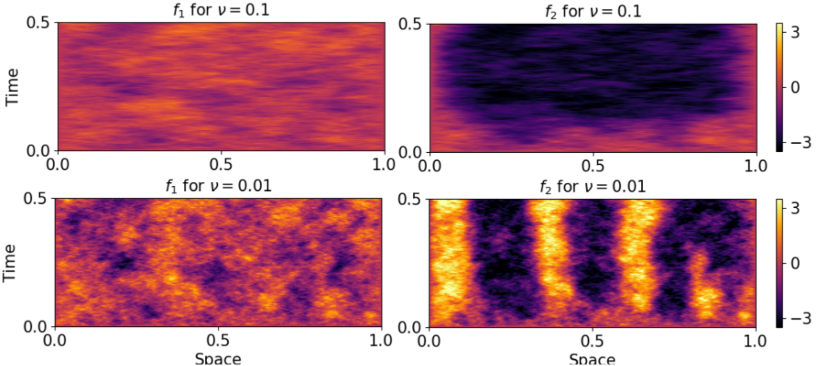
Other approaches:

$\sigma \to 0$: Ibragimov, Khasminskii (1999, 2000, 2001)

$T \to \infty$: Goldys, Maslowski (2002), Hildebrandt, Trabs (2023)

$\sigma, \nu \to 0$, nonparametric: Gaudlitz (2025)

Small diffusivity asymptotics



$$f_1(x) = -x(2 + \sin(x)), \qquad f_2(x) = -(x^3 - 9x) \; (\Phi^4_1)$$

Outline

What about stochastic heat equation?

Task: estimate the diffusivity $\vartheta > 0$ in

$$dX(t) = \vartheta \Delta X(t)\, dt + \sigma\, dW(t)$$

Problem: No Girsanov Theorem $\rightsquigarrow$ why?

Stochastic heat equation

$$dX(t) = \vartheta\Delta X(t)\,dt + \sigma\,dW(t)$$

- Laplace operator $\Delta : H^2(\Lambda) \to L^2(\Lambda)$ with Dirichlet or Neumann boundary condition on $\Lambda \subseteq \mathbb{R}^d$
- diffusivity constant $\vartheta > 0$
- space-time white noise $\dot{W}$, $\sigma > 0$

Spectral decomposition:
(λ_k, e_k) eigensystem of Laplace Δ with ONB (e_k), $\lambda_k \sim -k^{2/d}$.

$$X_k(t) := \langle X(t,\bullet), e_k\rangle_{L^2(\Lambda)} \Rightarrow dX_k(t) = \vartheta\lambda_k X_k(t)\,dt + \sigma\,dW_k(t),\ \ k \geqslant 1$$

with independent Brownian motions $(W_k)_{k\geqslant 1}$.

$\rightsquigarrow$ sequence of independent Ornstein-Uhlenbeck processes!

Spectral approach: identifiability

Observe $(X_k(t),\ t \in [0, T])$ continuously in time, where

$$\boxed{dX_k(t) = \vartheta\lambda_k X_k(t)dt + \sigma\, dW_k(t)}$$

Maximum-likelihood estimator (at frequency k):

$$\hat{\vartheta}_k := \frac{\int_0^T X_k(t)\, dX_k(t)}{\lambda_k \int_0^T X_k(t)^2 dt} = \vartheta + \frac{\int_0^T X_k(t)\sigma\, dW_k(t)}{\lambda_k \int_0^T X_k(t)^2 dt}$$

Asymptotics:

$$\boxed{\sqrt{T|\lambda_k|}\big(\hat{\vartheta}_k - \vartheta\big) \xrightarrow{T|\lambda_k|\to\infty} \mathcal{N}(0, 2\vartheta)}$$

Consequence: (T fixed)
The drift parameter is identifiable via $k \to \infty$ when observing continuously $(X(t,x), t \in [0, T], x \in \Lambda)$.

Spectral estimator

Observe the *Fourier modes* $(X_k(t),\ t \in [0,T])$ for $k = 1,\dots,K$, continuously in time, where

$$dX_k(t) = \langle e_k, dX(t)\rangle_{L^2(\Lambda)} = \vartheta\lambda_k X_k(t)dt + \sigma\, dW_k(t)$$

Maximum-likelihood estimator (up to frequency K):

$$\hat{\vartheta}_{1:K} := \frac{\sum_{k=1}^K \lambda_k \int_0^T X_k(t)\, dX_k(t)}{\sum_{k=1}^K \lambda_k^2 \int_0^T X_k(t)^2 dt}$$

Hübner, Rozovskii (1995):

$$\Big(T\sum_{k=1}^K |\lambda_k|\Big)^{1/2}\big(\hat{\vartheta}_{1:K} - \vartheta\big) \xrightarrow{T\sum_{k=1}^K |\lambda_k| \to \infty} \mathcal{N}(0, 2\vartheta)$$

and this is optimal among all estimators.

In particular, $\hat{\vartheta}_{1:K} = \vartheta + \mathcal{O}_P(K^{-\frac{1}{2}-\frac{1}{d}})$ for $K \to \infty$, T fixed.

Spectral method: extensions

Semi-linear SPDE:

$$dX(t) = \left(\vartheta\Delta X(t) + F(X(t))\right) dt + (-\Delta)^{-\gamma} dW(t)$$

Approximate MLE (P_K projection onto first K Fourier modes):

$$\hat{\vartheta}_{1:K} = \frac{\int_0^T \langle (-\Delta)^{1+2\alpha} P_K X(t), P_K(dX(t) - F(P_K X(t))dt)\rangle}{\int_0^T \|(-\Delta)^{1+2\alpha} P_K X(t)\|^2 dt}$$

Specific examples:

- Reaction-diffusion equation $F(X(t))(x) = f(X(t,x))$
- Burgers equation $F(X(t))(x) = -X(t,x)\partial_x X(t,x)$ in 1D
- 2D-stochastic Navier-Stokes equation
- Cahn-Hillard equation $dX(t)=(-\vartheta\Delta^2+\Delta(X(t)^3-X(t)))dt+BdW(t)$
- 2D-Navier Stokes equation

Cialenco, Glatt-Holtz (2011), Pasemann, Stannat (2020)

Spectral method: limitations

- Observation scheme:
  - When do we observe Fourier modes?
  - Approximation error due to (more realistic?) discrete observations?

- Do we know the eigenfunctions?
  - Δ on general domains?
  - Space-dependent diffusivity: $\Delta_\vartheta f(x) = \nabla \bullet (\vartheta(x) \nabla f(x))$ with $\vartheta : \Lambda \to \mathbb{R}^{d \times d}$ has ϑ-dependent eigenfunctions.

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Discrete observations in space and/or time

Observations: discrete, but asymptotically dense (high-frequence/in-fill asymptotics).

Idea (cf. volatility estimation):

- Determine p-variation of solution paths in time and/or space.
- Solve this for the parameter(s) of interest.
- Use discrete p-variation and form plug-in estimator.

e.g. Cialenco, Huang (2020), Hildebrand, Trabs (2023), Olivera, Tudor (2025) and references therein.

Local measurements

Typical features of space-time measurements:

- Data is discrete, obtained by local averaging.
- Data is corrupted by measurement noise.

Next:
Observe SPDE solution under spatial convolution with a known kernel (at resolution $\delta \to 0$).
Still, continuous in time and without measurement error.

Advantage: requires only weak SPDE solution!

Local measurements

Why intuitively is diffusivity identifiable?

The diffusivity in the drift grows with the frequency, while the white noise level remains constant.

$\rightsquigarrow$ signal-to-noise ratio grows in the frequency domain.

Question: Identifiability in spatial domain?

Spatial resolution δ of measurement around $x_0 \in \Lambda$:

$$X_\delta(t) := \int_\Lambda X(t,x) K_\delta(x - x_0)\,dx = (X(t,\bullet) * K_\delta)(x_0)$$

$K_\delta(x) = \delta^{-d/2} K(x/\delta)$ for K with compact support, $\|K\|_{L^2} = 1$

$$dX_\delta(t) = \vartheta(\Delta X(t,\bullet) * K_\delta)(x_0)dt + \sigma\, dW_\delta(t)$$

1D-Brownian motion $W_\delta(t) = \int_\Lambda K_\delta(x - x_0) dW(t,x)$.

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1D-Brownian motion $W_\delta(t) = \int_\Lambda K_\delta(x-x_0) dW(t,x)$.

Augmented MLE from local measurements

By partial integration

$$dX_\delta(t) = \vartheta\big(X(t,\bullet) * \Delta K_\delta\big)(x_0)dt + \sigma\, dW_\delta(t)$$

Augmented local measurements: $(X_\delta(t), (X * \Delta K_\delta)(t, x_0))_{0\leqslant t\leqslant T}$

Augmented MLE:

$$\hat{\vartheta}_\delta := \frac{\int_0^T (X * \Delta K_\delta)(t,x_0) dX_\delta(t)}{\int_0^T (X * \Delta K_\delta)^2(t,x_0)\, dt} = \vartheta + \frac{\int_0^T (X * \Delta K_\delta)(t,x_0)\sigma dW_\delta(t)}{\int_0^T (X * \Delta K_\delta)^2(t,x_0)\, dt}$$

Proposition.(Altmeyer, MR 2021)

$$\delta^{-1}\big(\hat{\vartheta}_\delta - \vartheta\big) \xrightarrow{\delta\to 0} \mathcal{N}\Big(0, \frac{2\vartheta}{T\|\nabla K\|_{L^2}^2}\Big)$$

ϑ is identifiable from local measurements $(X(t,x))_{0\leqslant t\leqslant T, |x-x_0|\leqslant\delta}$.

Augmented MLE from local measurements

By partial integration

$$dX_\delta(t) = \vartheta\big(X(t,\bullet) * \Delta K_\delta\big)(x_0)dt + \sigma\, dW_\delta(t)$$

Augmented local measurements: $(X_\delta(t), (X * \Delta K_\delta)(t, x_0))_{0\leqslant t\leqslant T}$

Augmented MLE:

$$\hat\vartheta_\delta := \frac{\int_0^T (X * \Delta K_\delta)(t,x_0)dX_\delta(t)}{\int_0^T (X * \Delta K_\delta)^2(t,x_0)\,dt} = \vartheta + \frac{\int_0^T (X * \Delta K_\delta)(t,x_0)\sigma dW_\delta(t)}{\int_0^T (X * \Delta K_\delta)^2(t,x_0)\,dt}$$

Proposition.(Altmeyer, MR 2021)

$$\delta^{-1}\big(\hat\vartheta_\delta - \vartheta\big) \xrightarrow{\delta\to 0} \mathcal{N}\Big(0, \frac{2\vartheta}{T\|\nabla K\|_{L^2}^2}\Big)$$

ϑ is identifiable from local measurements $(X(t,x))_{0\leqslant t\leqslant T, |x-x_0|\leqslant\delta}$.

Can we estimate space-dependent diffusivity?

- Fact:
  a spatially varying diffusivity influences $X(t,x)$ for all x.
- Hope:
  $X(t,x_0)$ is mainly influenced by $\{\vartheta(x) \mid |x-x_0| \leqslant \delta\}$.
- Idea:
  estimate $\vartheta(x_0)$ by $\hat{\vartheta}$ from local observations around x_0.

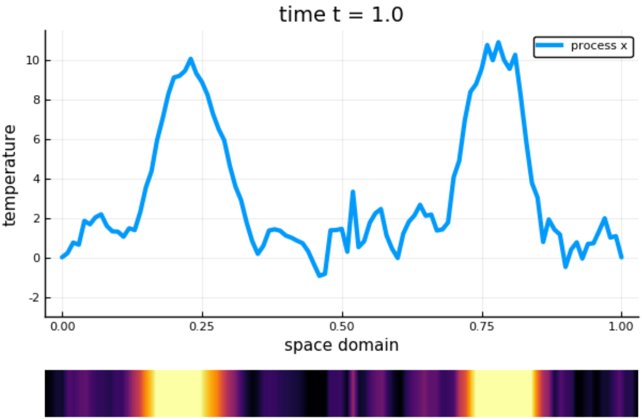
Stochastic heat equation with varying diffusivity

$$dX(t) = \Delta_\vartheta X(t)\, dt + \sigma\, dW(t)$$

- $\Delta_\vartheta g(x) := \sum_{i=1}^d \partial_{x_i} \vartheta(x) \partial_{x_i} g(x)$
- spatially varying diffusivity $\vartheta \in C^1(\Lambda)$, $\vartheta(x) > 0$
- space-time white noise $\dot{W}$, $\sigma > 0$

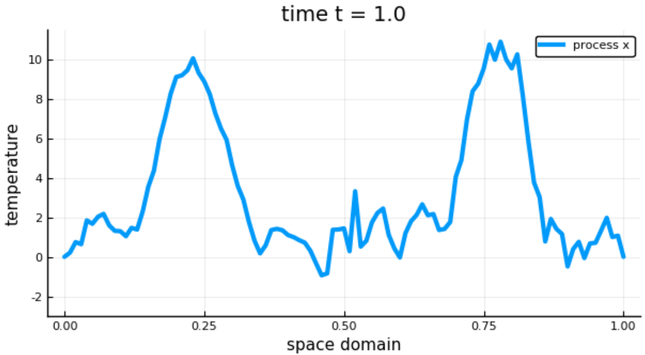
1D-Simulation

$$dX(t) = \Delta_\vartheta X(t)\, dt + \sigma\, dW(t)$$



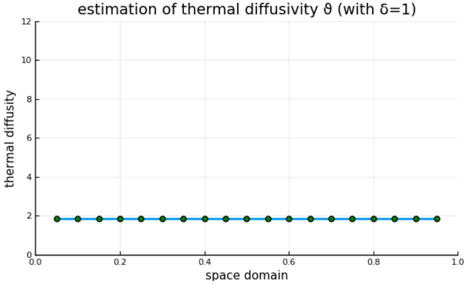
1D-Simulation

$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$



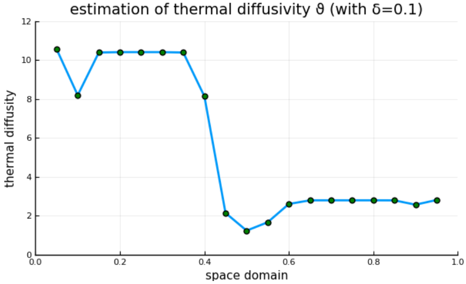
Estimator: $\delta = 1$

$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$



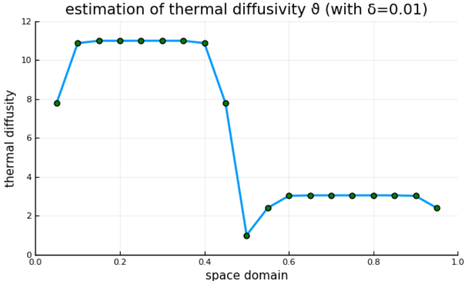
Estimator: $\delta = 0.1$

$$\boxed{\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)}$$



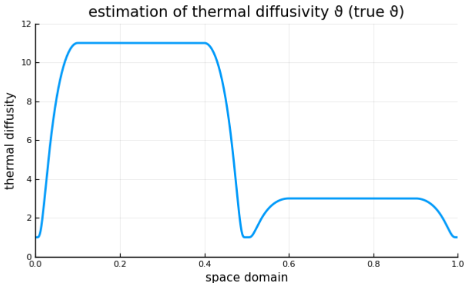
Estimator: $\delta = 0.001$

$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$



True function

$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$



Pointwise estimate of diffusivity

$$\boxed{\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)}$$

Idea: Augmented MLE $\hat{\vartheta}_\delta$ estimates $\vartheta(x)$ locally around x_0.

$$\hat{\vartheta}_\delta - \vartheta(x_0) = \underbrace{\mathcal{O}_P\left(\frac{\int(\vartheta(x_0+\delta\bullet) - \vartheta(x_0))|\nabla K|^2}{\vartheta(x_0)\|\nabla K\|_{L^2}^2}\right)}_{\text{"BIAS"}} + \underbrace{\mathcal{O}_P\left(\frac{\sqrt{\vartheta(x_0)}\delta}{\sqrt{T}\|\nabla K\|_{L^2}}\right)}_{\sqrt{\text{"VARIANCE"}}}$$

Remarks:

- For $\vartheta \in C^1(\Lambda)$ the bias is $\mathcal{O}_P(\delta)$.
- For $\vartheta \in C^1(\Lambda)$ and K radial-symmetric the bias is $o_P(\delta)$.
- The total error rate is then $\mathcal{O}_P(\delta)$.

General linear result I

$$dX(t,x) = A_\vartheta X(t,x)\,dt + \sigma(x)\,dW(t,x)$$

with second order operator $A_\vartheta = \Delta_\vartheta + \sum_i a_i(x)\partial_{x_i} + b(x)$.

Augmented MLE: $\hat{\vartheta}_\delta(x_0) := \frac{\int_0^T (X*\Delta K_\delta)(t,x_0)dX_\delta(t)}{\int_0^T (X*\Delta K_\delta)^2(t,x_0)\,dt}$

Theorem. (Altmeyer, MR 2021)
Under mild regularity conditions we have

$$\delta^{-1}\big(\hat{\vartheta}_\delta(x_0) - \vartheta(x_0)\big) \xrightarrow{\delta\to 0} \mathcal{N}\Big(\frac{\int \langle\nabla\vartheta(x_0),x\rangle|\nabla K(x)|^2 dx}{\|\nabla K\|_{L^2}^2}, \frac{2\|K\|_{L^2}^2}{T\|\nabla K\|_{L^2}^2}\Big)$$

General linear result II

Consequences and remarks:

- Robust to lower order drift coefficients $a(x), b(x)$ and noise level $\sigma(x)$
- $\vartheta(x_0)$ is nonparametrically identifiable
- Estimator can be applied at different locations x_i separately
- Confidence intervals with Gaussian quantiles
- Convergence rate δ is minimax-optimal for any estimator

Proof ingredients:

- Scaling via $X(\delta^2 t, \delta x)$, $t \in [0, \delta^{-2} T]$, $x \in \delta^{-1}\Lambda$
- Heat kernel and semigroup bounds (via Feynman-Kac)
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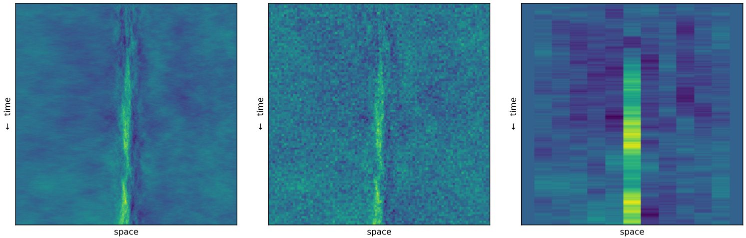
Dynamic versus static noise in SPDEs

SPDE observations under measurement errors: Observe Y, the SPDE solution X corrupted by space-time white noise $\dot{V}$:

$$\dot{Y}(t,x) = X(t,x) + \varepsilon \dot{V}(t,x), \quad t \in [0,1],\ x \in \Lambda,$$
$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$

with *static noise level* $\varepsilon > 0$ and *dynamic noise level* $\sigma > 0$.



Estimation of $\vartheta(x_0)$

Space-time smoothing kernel:

Let $K \in C^\infty([0,1] \times \mathbb{R}^d)$, $\|K\|_{L^2} = 1$, $\operatorname{supp}(K) \subseteq [0,1]^{d+1}$ and

$$K^\delta_{t_0,x_0}(t,x) = \delta^{-(d+2)} K(\delta^{-2}(t-t_0), \delta^{-1}(x-x_0))$$

Note: $\operatorname{supp}(K^\delta_{t_0,x_0}) \subseteq [t_0, t_0+\delta^2] \times (x_0 + [0,\delta]^d)$.

Plug-in approach:

- Use noisy observation $\langle \varphi, \dot{Y} \rangle$ as proxy for $\langle \varphi, X \rangle$
  $\rightsquigarrow$ for $\varphi = -\partial_t K^\delta_{t_0,x_0}$ requires $\delta \gtrsim (\varepsilon/\sigma)^{1/2}$.
  $\rightsquigarrow$ optimal choice $\delta \sim (\varepsilon/\sigma)^{1/2}$.
- For $\vartheta \in C^\alpha(\Lambda)$, $\alpha \geqslant 1$, profit from estimating $\vartheta(x)$ for $|x - x_0| \leqslant h$, $h \gtrsim \delta$ for x from a grid $\mathcal{X}_\delta = \Lambda \cap \delta\,\mathbb{Z}^d$.

Final estimator for $\vartheta(x_0)$

Intuition: regress $\dot{X}(t, x_0)$ onto $\Delta X(t, x_0)$.

$$\hat{\vartheta}(x_0) = \frac{\sum_{i=1}^{\delta^{-2}-1} \sum_{x \in \mathcal{X}_\delta, |x-x_0| \leqslant h} \langle -\partial_t K^\delta_{i\delta^2,x}, \dot{Y}\rangle \langle \Delta K^\delta_{(i-1)\delta^2,x}, \dot{Y}\rangle}{\sum_{i=1}^{\delta^{-2}-1} \sum_{x \in \mathcal{X}_\delta, |x-x_0| \leqslant h} \langle \Delta K^\delta_{i\delta^2,x}, \dot{Y}\rangle \langle \Delta K^\delta_{(i-1)\delta^2,x}, \dot{Y}\rangle}$$

Theorem. (Pasemann, MR 2024)
For $\vartheta \in C^\alpha(\Lambda)$, $\delta \sim (\varepsilon/\sigma)^{1/2}$, $h \sim (\varepsilon/\sigma)^{\frac{d+2}{4\alpha+2d}}$, $d \geqslant 3$, $\alpha \in [1, 2 - 2/(d+4)]$, we obtain

$$\hat{\vartheta}(x_0) - \vartheta(x_0) = \mathcal{O}_P\Big(\big(\tfrac{\varepsilon}{\sigma}\big)^{\alpha(1+d/2)/(2\alpha+d)}\Big)$$

Lower bound: Hellinger distance for Gaussian laws

Theorem. The Hellinger distance between cylindrical Gaussian measures on a separable Hilbert space satisfies

$$H(\mathcal{N}_{cyl}(0,Q_0),\mathcal{N}_{cyl}(0,Q_1)) \leqslant \tfrac{1}{2}\|Q_0^{-1/2}Q_1^{1/2}-(Q_1^{-1/2}Q_0^{1/2})^*\|_{HS}$$

provided $\operatorname{ran}(Q_0^{1/2})=\operatorname{ran}(Q_1^{1/2})$ and Q_0, Q_1 are one-to-one.

Remark. Interpreting $Q_1-Q_0:\operatorname{ran}(Q_i^{-1/2})\to\operatorname{ran}(Q_i^{1/2})$ we may write

$$H(\mathcal{N}_{cyl}(0,Q_0),\mathcal{N}_{cyl}(0,Q_1)) \leqslant \tfrac{1}{2}\|Q_0^{-1/2}(Q_1-Q_0)Q_1^{-1/2}\|_{HS}$$

Covariance for stochastic evolution equations

Stochastic evolution equation under measurement errors:

$$Y(t,x) = X(t,x) + \varepsilon \dot{V}(t,x),$$
$$\dot{X}(t,x) = A_\vartheta X(t,x) + \sigma \dot{W}(t,x), \quad X_0 = 0$$

with normal generators $A_\vartheta : \mathrm{dom}(A_\vartheta) \subseteq H \to H$, $\vartheta \in \{0,1\}$, and $\mathrm{dom}(A_0) = \mathrm{dom}(A_1)$.

Proposition. $Y \sim \mathcal{N}_{cyl}(0, Q_\vartheta)$ on $L^2([0,T];H)$ with

$$\boxed{Q_\vartheta = \varepsilon^2 \,\mathrm{Id} + \sigma^2 S_\vartheta S_\vartheta^* \quad \text{with} \quad S_\vartheta f(t) = \int_0^t e^{A_\vartheta (t-s)} f(s)\, ds}$$

Key formula: $S_1 - S_0 = S_0(A_1 - A_0)S_1$ on $L^2([0,T];H)$.

Hellinger bound for stochastic evolution equations

Stochastic evolution equation under measurement errors:

$$Y(t,x) = X(t,x) + \varepsilon \dot{V}(t,x),$$
$$\dot{X}(t,x) = A_\vartheta X(t,x) + \sigma \dot{W}(t,x), \quad X_0 = 0$$

Theorem. If the spectrum of A_ϑ has negative real part, then for the Hellinger distance between the laws of Y for $\vartheta \in \{0,1\}$

$$H(\mathcal{N}_{cyl}(0,Q_0), \mathcal{N}_{cyl}(0,Q_1)) \leqslant$$
$$T\Big(\|(\tfrac{\varepsilon^2}{\sigma^2} R_1^2 + \mathrm{Id})^{-1/2}(A_1 - A_0)(\mathrm{Id} - 2TR_0)^{-1/2}(\tfrac{\varepsilon^2}{\sigma^2} R_0^2 + \mathrm{Id})^{-1/2}\|_{HS(H)}$$
$$+ \|(\tfrac{\varepsilon^2}{\sigma^2} R_0^2 + \mathrm{Id})^{-1/2}(A_1 - A_0)(\mathrm{Id} - 2TR_1)^{-1/2}(\tfrac{\varepsilon^2}{\sigma^2} R_1^2 + \mathrm{Id})^{-1/2}\|_{HS(H)} \Big)$$

holds with real parts $R_i = (A_i + A_i^*)/2$. (Pasemann, MR 2025)

Parametric lower bounds

Minimax lower bounds for parameter $\vartheta > 0$:

process	rate
$dX_t = -\vartheta X_t dt + dW_t$	$T^{-1/2}(\varepsilon^2\vartheta^2 + 1)\vartheta^{1/2}$
$\dot{X}(t,x) = \vartheta\Delta X(t,x) + \dot{W}(t,x)$	$T^{-1/2}\varepsilon^{(d+2)/4}$
$\dot{X}(t,x) = \nu\Delta X(t,x) + \vartheta\partial_\xi X(t,x) + \dot{W}(t,x)$	$T^{-1/2}\nu^{(d+2)/4}\varepsilon^{d/4}$
$\dot{X}(t,x) = \nu\Delta X(t,x) - \vartheta X(t,x) + \dot{W}(t,x)$	$T^{-1/2}\nu^{d/4}\varepsilon^{(d-2)_+/4}$

Remarks:

- dynamic noise: for σdW_t: $\varepsilon \rightsquigarrow \varepsilon/\sigma$!
- OU process: no impact of ε for ϑ fixed.
- Diffusivity: dim. $d \leqslant 5$; identifiable for $T \to \infty$ or $\varepsilon \to 0$
- Transport: $d \leqslant 7$; ident.: $T \to \infty$ or $\nu \to 0$ or $\varepsilon \to 0$
- Source: $d \leqslant 9$ ($d = 2$ has factor $(\log \varepsilon^{-1})^{-1/2}$); identifiable for $T \to \infty$ or $\nu \to 0$ or (if $d \geqslant 2$) $\varepsilon \to 0$

Nonparametric lower bounds

Minimax lower bounds for $\vartheta(x_0) > 0$, $\vartheta \in C^\alpha(\Lambda)$:

process	rate
$\dot X(t,x) = \Delta_\vartheta X(t,x) + \dot W(t,x)$	$T^{-\frac{\alpha}{2\alpha+d}}\varepsilon^{\frac{\alpha(d+2)}{4\alpha+2d}}$
$\dot X(t,x) = \Delta X(t,x) + \partial_\xi(\vartheta(x)X(t,x)) + \dot W(t,x)$	$T^{-\frac{\alpha}{2\alpha+d}}\varepsilon^{\frac{\alpha d}{4\alpha+2d}}$
$\dot X(t,x) = \Delta X(t,x) - \vartheta(x)X(t,x) + \dot W(t,x)$	$T^{-\frac{\alpha}{2\alpha+d}}\varepsilon^{\frac{\alpha(d-2)_+}{4\alpha+2d}}$

Remarks on diffusivity estimation:

- Rate holds for dimension $d \leqslant 5$ and $T \leqslant \varepsilon^{1-\alpha}$.
- For $\alpha \geqslant 1$, T fixed: rate $\varepsilon^{\frac{\alpha(d+2)}{4\alpha+2d}}$ is upper bound.
- For $\alpha < 1$, T fixed: lower bound becomes $\varepsilon^{\frac{5\alpha}{4\alpha+6}}$.
  $\rightsquigarrow$ *Ellbow effect*: rate slows down for $d \in \{1, 2\}$ and accelerates for $d > 4$!

Q & A

Thanks a lot for your attention!

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